CHAPTER 5
MULTIPLE INTEGRALS

◙DOUBLE INTEGRATION ON A GENERAL REGION►

The double integral of the function $f(x,y)$ on the region R in the xy – plane is defined as:

$$I = \iint_R f(x,y)\,dA \qquad (1)$$

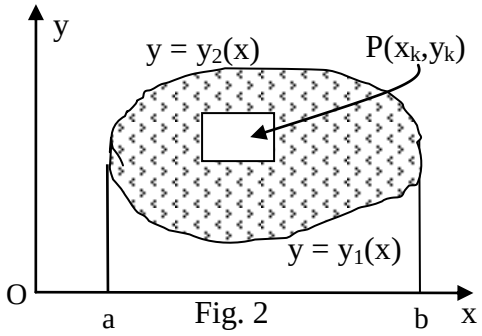
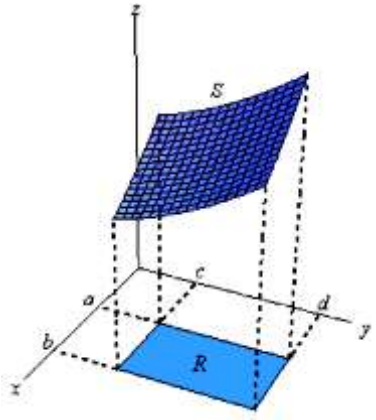


Fig. 2

To calculate the integral, we assume the region R is bounded by the curves $y = y_1(x)$, $y = y_2(x)$, and the ordinates x = a, x = b as in Fig.2. Divide the region into small elements in the shape of small rectangles with dimensions $\delta x, \delta y$ as in Fig.2.

Let $P(x_k, y_k)$ any point in the element of area $\delta A_k = \delta x_k \delta y_k$ and forming the product $f(x_k, y_k)\delta A_k$.

Forming a vertical strip with width δx from the curve $y = y_1(x)$ to the curve $y = y_2(x)$.



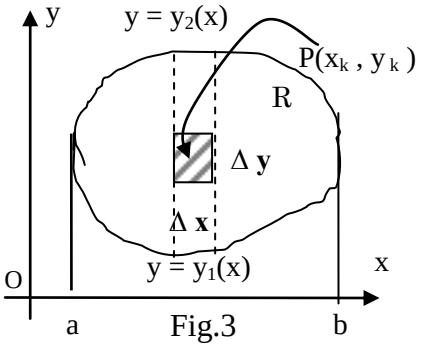


Fig.3

Adding all such elements in y – direction as in Fig.3 , we obtain

$$\left[\sum_{y=y_1}^{y_2} f(x_k, y_k)\delta y_k\right]\delta x$$

Note that the resulting summation is a function of x. Then adding all such vertical strips in the x-direction from x = a to x = b , we get

$$\sum_{x=a}^{b}\left[\sum_{y_1(x)}^{y_2(x)} f(x_k, y_k)\delta y_k\right]\delta x_k$$

Take the limit as $\delta x, \delta y \to 0$, the resulting summation will be transformed into an integration in the form

$$\lim_{\delta A\to 0}\sum_x\sum_y f(x_k, y_k)\delta y_k \delta x_k = \int_{x=a}^{b}\int_{y_1(x)}^{y_2(x)} f(x,y)\,d\,y\;d\,x$$

This is called the double integration of the function $f(x,y)$ on the region R.Note that in the double integral

$$I = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x,y)\, d\, y \right] d\, x$$

(i) We work from the inside integral.

(ii) We integrate w.r.t. x when the limits are values of x.

(iii) We integrate w.r.t. y when the limits are values of y.

NOTE►

The double integral can also be calculated first with respect to x , the substituting the limits of integration in terms of y. We thus obtain a function of y which can be integrated with respect to y to obtain the result.

$$I = \iint_A f(x,y)dxdy = \int_C^D d\, y \left[\int_{x=x_1(y)}^{x=x_2(y} f(x,y)\ d\, x \right]$$

Also note that

$$I = \iint_A f(x,y)dxdy = \iint f(x,y)dydx$$

EXAMPLE 1► Calculate

$$\int_2^4 \int_1^2 (x+2y)dydx \text{ and } \int_1^2 \int_2^4 (x+2y)dxdy ,$$

and compare the results.

SOLUTION

(x + 2 y) is first integrated w.r.t. x between x = 2 and x = 4, with y regarded as constant for the time being.

$$I=\int_1^2\left[\int_2^4(x+2y)dx\right]dy=\int_1^2\left[\frac{x^2}{2}+2yx\right]_2^4dy$$

$$=\int_1^2\left[(8+8y)-(2+4y)\right]dy=\int_1^2\left[6+4y\right]dy$$

$$=\left[6y+2y^2\right]_1^2=(12+8)-(6+2)=12$$

Then we reverse the order of integration, We first integrate with respect to y, then with respect to x , the integral takes the form:

$$I=\int_2^4\int_1^2(x+2y)dydx=12$$

It is clear that:

$$\int_2^4\int_1^2(x+2y)dydx=\int_1^2\int_2^4(x+2y)dxdy$$

EXAMPLE 2► Evaluate each of the following integrals over the given region D.

$(i)\ \iint_D e^{x/y}\,dA$, $D=\left\{(x,y):1\le y\le 2, y\le x\le y^3\right\}$

$(ii)\ \iint_D\left(4xy-y^3\right)dA$, D is the region bounded by

$y=\sqrt{x}$ and $x=y^3$

$(iii)\ \iint_D\left(6x^2-40y\right)dA$, D is triangle with vertices

$(0,3),(1,1),(5,3)$.

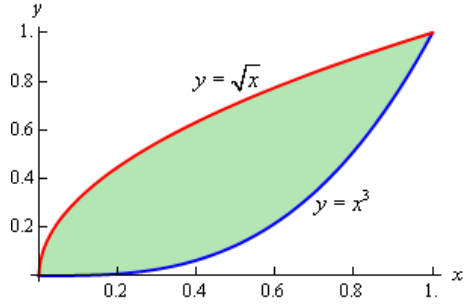
SOLUTION

$(i)\ \iint_D e^{x/y}\,dA\ ,\ D=\left\{(x,y):1\le y\le 2, y\le x\le y^3\right\}$

$$\iint_D e^{x/y}\,dA=\int_1^2\int_y^{y^3} e^{x/y}\,dx\,dy=\int_1^2 dy\times y\left[e^{x/y}\right]_y^{y^3}=\int_1^2 dy\times y\left[e^{y^2}-e\right]$$

$$=\frac{1}{2}\int_1^2 2ye^{y^2}dy-e\int_1^2 y\,dy=\frac{1}{2}\left[e^{y^2}-y^2e\right]_1^2=\frac{1}{2}\left(e^4-4e\right)$$

$(ii)\ \iint_D\left(4xy-y^3\right)dA$, D is the region bounded by



$$y=\sqrt{x}\ \text{and}\ y=x^3$$

The points of intersection are

$$y^2=x=y^3\Rightarrow y^2(1-y)=0$$

$$y=0\ \text{and}\ y=1$$

$$\Rightarrow x=0\ \text{and}\ x=1$$

$$(0,0)\ \text{and}\ (1,1)$$

$$(ii)\ \iint_D\left(4xy-y^3\right)dA=\int_0^1 dx\int_{x^3}^{\sqrt{x}}\left(4xy-y^3\right)dy$$

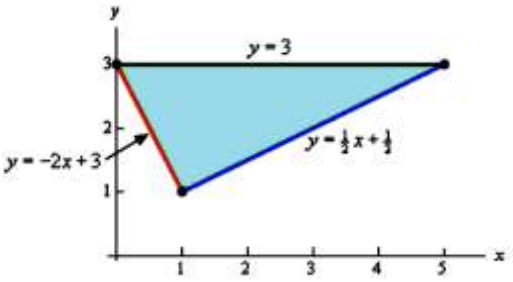
$$=\int_0^1 dx\left(2xy^2-\frac{1}{4}y^4\right)_{x^3}^{\sqrt{x}}=\int_0^1\left(2x\left(x-x^6\right)-\frac{1}{4}\left(x^2-x^{12}\right)\right)dx$$

$$=\int_0^1\left(\frac{7}{4}x^2-x^7+\frac{1}{4}x^{12}\right)dx=\left(\frac{7}{12}x^3-\frac{x^8}{8}+\frac{x^{13}}{52}\right)_1^2=\frac{55}{156}$$

$(iii)\ \iint_D \left(6x^2 - 40y\right)dA$,

D is triangle with vertices $(0,3),(1,1),(5,3)$.

Since we have two points on each edge it is easy to get the equations for each edge .



The region is bounded by the lines

$$y = 3\ ,\ y = -2x + 3\ and\ \ y = \frac{1}{2}x + \frac{1}{2}$$

Solving for x, we get

$$x_1 = -\frac{1}{2}y + \frac{3}{2}\ ,\quad and\ \ x_2 = 2y - 1$$

$$I = \iint_D \left(6x^2 - 40\right)dA = \int_1^3 dy \int_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} \left(6x^2 - 40\right)dx$$

$$I = \int_1^3 dy \left[2x^3 - 40xy\right]_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1}$$

$$I = 2\int_1^3 dy \left[(2y-1)^3 - \left(-\frac{1}{2}y + \frac{3}{2}\right)^3\right]$$

$$-40\int_1^3 dy \left[y(2y-1) - y\left(-\frac{1}{2}y + \frac{3}{2}\right)\right]$$

$$I = \int_1^3 dy \left[100y - 100y^2 + 2(2y-1)^3 - 2\left(-\frac{1}{2}y + \frac{3}{2}\right)^3\right]$$

$$I = \left[50y^2 - \frac{100}{3}y^3 + \frac{1}{4}(2y-1)^4 + \left(-\frac{1}{2}y + \frac{3}{2}\right)^4 \right]_1^3 = -\frac{935}{3}$$

EXAMPLE 3►

Calculate $\int_0^{\pi} \int_0^{x} x \sin y \, dy \, d x$

SOLUTION

$$I = \int_0^{\pi} x \left(\int_0^{x} \sin y dy \right) d x$$

Assume x as a constant, and integrating w.r.t. y . We get:

$$I = \int_0^{\pi} \left[-x \cos y \right]_0^{x} dx = \int_0^{\pi} \left[-x \cos x + x \cos(0) \right] dx$$

$$= \int_0^{\pi} \left(x - x \cos x \right) dx = \frac{x^2}{2} \Big]_0^{\pi} - \int_0^{\pi} x \cos x \, d x = \frac{\pi^2}{2} - I'$$

The second integral I' on LH.S.can be integrated by parts:

$$I' = \int_0^{\pi} x \cos x . dx$$

$$u = x \ \rightarrow dv = \cos x \, dx$$

$$\ddots$$

$$\searrow$$

$$du = dx \ \leftarrow v = \sin x$$

$$I' = x \sin x - \int \sin x dx = x \sin x + \cos x$$

$$\therefore I = \frac{\pi^2}{2} - \left(x \sin x + \cos x \right)_0^{\pi}$$

$$I = \frac{\pi^2}{2} - \left[(0-1) - (1)\right] = \frac{\pi^2}{2} + 2$$

◙CHANGE THE ORDER OF INTEGRATION►

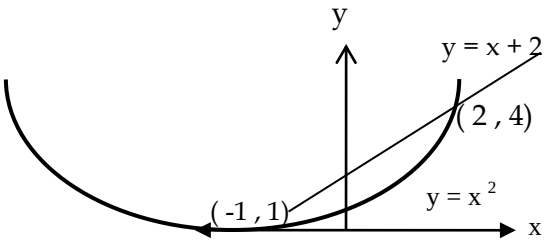
If a region is bounded from the left by x = $h_1(y)$ and the right by x = $h_2(y)$ and below and above by y = c and y = d, then we can find the double integral of "dxdy" by first integrating with respect to x then with respect to y. Sometimes there is a choice to make as to whether to integrate first with respect to x and then with respect to y. We do whatever is easier.

EXAMPLE 1►

Calculate the area bounded by the parabola $y = x^2$, and the line y = x + 2, reverse the order of integration, and calculate again.

SOLUTION

At first we must obtain the intersection points:



$x^2 = x + 2$ **=>**

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

=> $x_1 = -1$, $x_2 = 2$

Substitute into : y = x + 2 , we get:

$x_1 = -1 \Rightarrow y_1 = -1 + 2 = 1$ **=>** (-1, 1)

$x_2 = 2 \Rightarrow y_2 = 2 + 2 = 4 \Rightarrow (2, 4)$

$$A = \iint dydx$$

$$A = \int_{-1}^{2} dx \int_{x^2}^{x+2} dy = \int_{-1}^{2} (x + 2 - x^2) dx = \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big|_{-1}^{2}$$

$$A = \frac{1}{2}(4-1) + 2(2+1) - \frac{1}{3}(8+1) = \frac{3}{2} + 6 - 3 = \frac{9}{2} = 4.5$$

Note that: when calculate the integral by using the formula

$$A = \iint dxdy$$

We must divide the region of integration into two portions

$A_1 : 0 \le y \le 1$ And $A_2 : 1 \le y \le 4$

The result will be the sum of the two results , as follows:

$$A = \int_0^1 dy \int_{-\sqrt{y}}^{\sqrt{y}} dx + \int_1^4 dy \int_{y-2}^{\sqrt{y}} dx$$

$$= \int_0^1 (\sqrt{y} + \sqrt{y}) dy + \int_1^4 (\sqrt{y} - y + 2) dy$$

$$= \frac{2(y)^{3/2}}{3/2}\Big|_0^1 + \left[\frac{y^{3/2}}{3/2} - \frac{y^2}{2} + 2y\right]_1^4$$

$$= \frac{4}{3}(1-0) + \frac{2}{3}(8-1) - \frac{1}{2}(16-1) + 2(4-1)$$

$$= \frac{4}{3} + \frac{14}{3} - \frac{15}{2} + 6 = 12 - 7.5 = 4.5$$

EXAMPLE 2►
Evaluate the following integrals by first reversing the order of integration

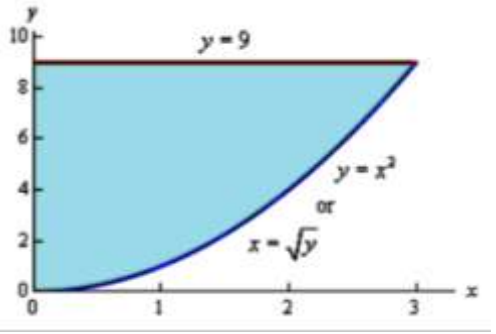
$$(i)\int_0^3\int_{x^2}^9 x^3 e^{y^3}\,dy\,dx \quad (ii)\int_0^8\int_{\sqrt[3]{y}}^2 \sqrt{1+x^4}\,dx\,dy$$

SOLUTION
In order to reverse the order of integration.
* First sketch the region given by the original limits of integration.

* From the integral we see that the inequalities that define this region are,

$$0 \le x \le 3,\ x^2 \le y \le 9$$

These inequalities tell us that we want the region with $y = x^2$ on the lower boundary and $y = 9$ on the upper boundary that lies between $x = 0$ and $x = 3$.



The integral, with the order reversed, is now

$$(i)\int_0^3\int_{x^2}^9 x^3 e^{y^3}\,dy\,dx$$

$$I = \int_0^3\int_{x^2}^9 x^3 e^{y^3}\,dy\,dx = \int_0^9\int_0^{\sqrt{y}} x^3 e^{y^3}\,dx\,dy$$

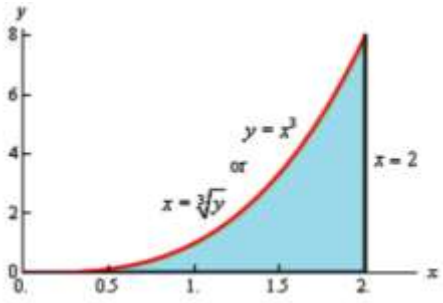
$$= \int_0^9 e^{y^3}\,dy\left[\frac{x^4}{4}\right]_0^{\sqrt{y}} = \frac{1}{4}\int_0^9 y^2 e^{y^3}\,dy$$

$$= \frac{1}{4\times 3}\int_0^9 3y^2 e^{y^3} dy = \frac{1}{12}\left[e^{y^3}\right]_0^9 = \frac{1}{12}\left[e^{729} - 1\right]$$

$$(ii)\ \int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{1+x^4}\, dx\, dy$$

The region is bounded by

$$\sqrt[3]{y} \le x \le 2,\ \ 0 \le y \le 8$$



Reversing the order of integration yields

$$0 \le y \le x^3 \ \ and \ \ 0 \le x \le 2$$

$$I = \int_0^2 \sqrt{1+x^4}\, dx \int_0^{x^3} dy$$

$$= \int_0^2 x^3\sqrt{1+x^4}\, dx = \frac{1}{4}\int_0^2 \sqrt{1+x^4}\,\left(4x^3\right)dx$$

$$= \frac{1}{4}\times\frac{\left(1+x^4\right)^{3/2}}{3/2}\Bigg]_0^2 = \frac{1}{6}\left[(1+16)^{3/2} - (1)^{3/2}\right] = \frac{69.1}{6} = 11.5$$

EXAMPLE 3► Evaluate the integral

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$$

SOLUTION

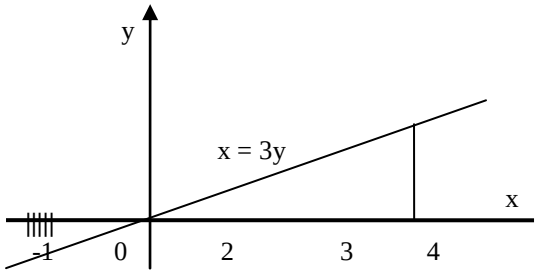
Try as you may, you will not find an anti-derivative of e^{x^2} and we don't want to get into power series expansions. We have another choice.We can change the order of integration.

The region is bounded above and below by y = 1/3 x and y = 0. The Fig. below shows the region.

The double integral with respect to y first and then with respect to x is

$$\int_0^3 \int_0^{x/3} e^{x^2}\, dy\, dx$$

The integrand is just a constant with respect to y so we get



$$\int_0^3 \left[e^{x^2} y \right]_0^{x/3} dx = \int_0^3 \left[\frac{x}{3} e^{x^2} \right] dx$$

This integral can be performed with simple u-substitution.

$$u = x^2 \qquad du = 2x\, dx$$

and the integral becomes

$$I = \int_0^9 \left[\frac{1}{6} e^u \right] du = \left[\frac{1}{6} e^u \right]_0^9 = \frac{1}{6}\left[e^9 - 1 \right]$$

EXAMPLE 4►

Draw the region of integration for the following integral

$$\int_0^8 \int_x^{\sqrt{8x}} xy \; dy \; dx$$

Calculate the integral , then change the order of integration and calculate its value again.

SOLUTION

The limits of y: from y = x

to $y = \sqrt{8x}$, i.e.

$$y : \underbrace{y = x}_{line} \Rightarrow \underbrace{y^2 = 8x}_{parabola}$$

$$I = \int_0^8 x dx \int_x^{\sqrt{8x}} y dy$$

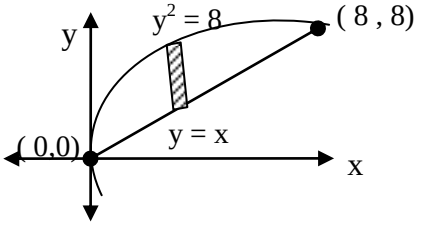
$$= \int_0^8 x dx \left(\frac{y^2}{2}\right)\Big|_x^{\sqrt{8x}}$$

$$= \frac{1}{2}\int_0^8 x(8x - x^2)dx$$

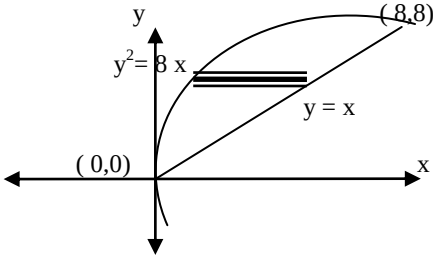
$$I = \frac{1}{2}\int_0^8 (8x^2 - x^3)dx$$

$$= \frac{1}{2}\left[\frac{8x^3}{3} - \frac{x^4}{4}\right]_0^8$$

$$= \frac{1}{2}\left[\frac{(8)^4}{3} - \frac{(8)^4}{4}\right] = \frac{(8)^4}{24} = \frac{512}{3} = 170.67$$



In order to change the order of integration we divide the region of integration by using slides parallel to x-axis . Thus, the limits of variation of x in terms of y will be:



$$\text{from } x = \frac{y^2}{8} \text{ to } x = y.$$

Then we integrate with respect to y from y = 0 to y = 8.

$$I = \int_0^8 \int_{y^2/8}^{y} xy \, d\,x d\,y$$

$$= \int_0^8 y dy \int_{y^2/8}^{y} x dx = \frac{1}{2}\int_0^8 [x^2]_{y^2/8}^{y} y dy$$

$$= \frac{1}{2}\int_0^8 y(y^2 - \frac{y^4}{64})dy = \frac{1}{2}\int_0^8 (y^3 - \frac{y5}{64})dy = \frac{1}{2}\left[\frac{y^4}{4} - \frac{y^6}{384}\right]_0^8$$

$$= \left[\frac{(8)^4}{8} - \frac{(8)^6}{768}\right] = \frac{512}{3} = 17.66 \blacksquare$$

EXERCISE 4.1

◙Calculate the following integrals:

1► $\int_0^{\pi} \int_0^{x} x \sin y \; dydx$

2► $\int_1^{\ell n 8} \int_1^{\ell n y} e^{x+y} \; dydx$

3► $\int_0^{\pi}\int_0^{\sin x} ydydx$ 4► $\int_1^{2}\int_y^{y^2} dxdy$

◙ In each of the following integrals draw the region of integration, then calculate the integral. Change the order of integration then calculate the integral again.

5► $\int_0^{2}\int_1^{e^x} dydx$

6► $\int_0^{\sqrt{2}}\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} ydxdy$

7► $\int_{-2}^{1}\int_{x^2+4x}^{3x+2} dydx$ 8► $\int_0^{1}\int_{\sqrt{y}}^{1} dxdy$

◙ In each of the following use the double integration to calculate the area of the region enclosed by the given curves:

9► $y^2 + x = 0$, y = x + 2

10► $x = y^2$, $x = 2y - y^2$

11► $y = \sqrt{a^2 - x^2}$, $x = \pm a$ and y = - a

12► $x = y - y^2$, x + y = 0

Calculate the following integrals:

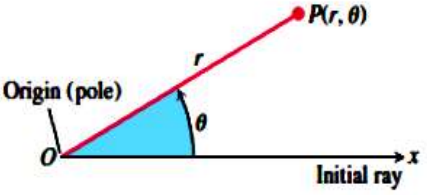
13► $\int_0^{3}\int_{-x}^{x(2-x)} dydx$ 14► $\int_0^{1}\int_y^{\sqrt{y}} dxdy$

15► $\int_{-1}^{2}\int_{y^2}^{y+2} dxdy$ 16► $\int_{-1}^{0}\int_{-2x}^{1-x} dydx + \int_0^{2}\int_{-x/2}^{1-x} dydx.$

◙ AREA IN POLAR COORDINATES►

Definition of Polar Coordinates (r , θ)

To define polar coordinates, we first fix an origin O (called the pole) and an initial ray from O . Then each point P can belocated by assigning to it a polar coordinate pair (r , θ)in which r gives the directed distance from O to P and gives the directed angle from the initial ray to ray OP.



Equations Relating Polar and Cartesian Coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$

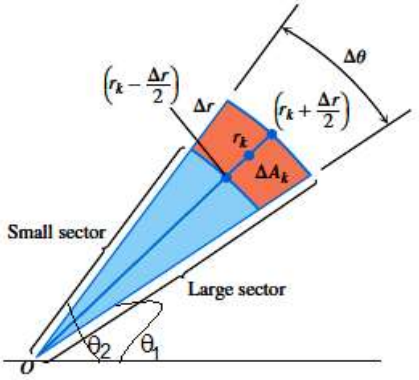
To find the area of the plane figure bounded by the polar curve $r = f(\theta)$, and the radius vectors at $\theta = \theta_1$ and $\theta = \theta_2$.

We proceed very much as before. We find that the Area of the element

$\delta A = \delta r . r\,\delta\theta$,

i.e. in general

$$A = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} r\, dr\, d\theta$$

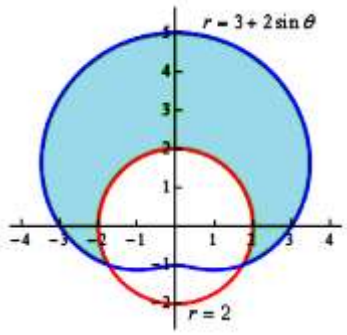


$$\Delta A_k = \begin{pmatrix}\text{area of}\\ \text{large sector}\end{pmatrix} - \begin{pmatrix}\text{area of}\\ \text{small sector}\end{pmatrix}$$

leads to the formula $\Delta A_k = r_k\,\Delta r\,\Delta\theta$.

EXAMPLE1►

Determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.



SOLUTION

The points of intersection of the two curves

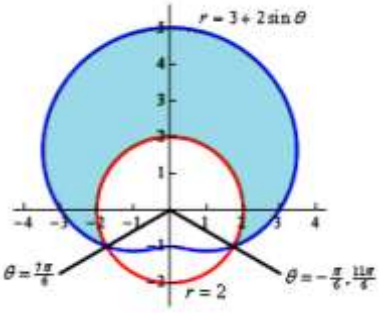
$$3 + 2\sin\theta = 2 \Rightarrow \sin\theta = -1/2$$

$$\Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

The region of integration can be determined by

$$-\frac{\pi}{6} \le \theta \le \frac{7\pi}{6},$$

$$2 \le r \le 3 + 2\sin\theta$$



$$A = \int_{-\pi/6}^{7\pi/6} \int_{2}^{3+2\sin\theta} r\,dr\,d\theta$$

$$A = \frac{1}{2}\int_{-\pi/6}^{7\pi/6} d\theta \left[r^2 \right]_2^{3+2\sin\theta}$$

$$= \frac{1}{2}\int_{-\pi/6}^{7\pi/6} \left[5 + 12\sin\theta + 4\sin^2\theta \right] d\theta$$

$$= \frac{1}{2}\int_{-\pi/6}^{7\pi/6} \left[7 + 12\sin\theta - 2\cos 2\theta \right] d\theta$$

$$= \frac{1}{2}\left[7\theta - 12\cos\theta - \sin 2\theta \right]_{-\pi/6}^{7\pi/6}$$

$$= \frac{1}{2}\left[\frac{28\pi}{3} - 12\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right) - \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \right) \right]$$

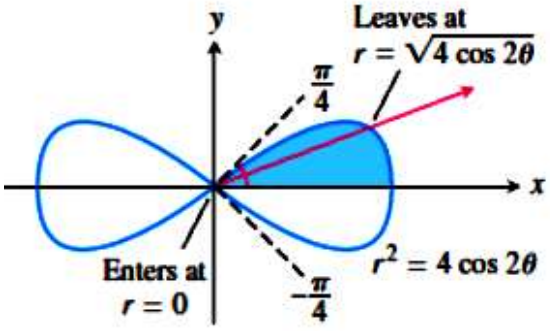
$$A = \frac{14\pi}{3} + \frac{11\sqrt{3}}{2} = 24.187$$ ■

EXAMPLE 2►

Calculate the area enclosed by the luminescent $r^2 = 4\cos 2\theta$

SOLUTION

From the symmetry we can calculate the area located in the first quadrant only and multiply the result by 4.



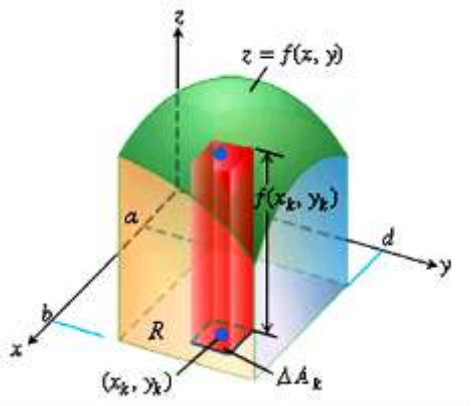
$$A = 4 \times \int_0^{\pi/4} \int_0^{\sqrt{4\cos 2\theta}} r\, d\, r\, d\theta$$

$$A = 4 \times \int_0^{\pi/4} \left(\frac{r^2}{2} \right)_0^{\sqrt{4\cos 2\theta}} d\theta = 4 \times \frac{1}{2} \int_0^{\pi/4} 4\cos 2\theta\, d\theta$$

$$= 8 \times \left[\frac{1}{2} \sin 2\theta \right]_0^{\pi/4} = 8 \times \frac{1}{2}(1-0) = 4$$

◙ CALCULATING THE VOLUME BY USING DOUBLE INTEGRATION►

Assuming that the volume is bounded from above by a surface whose equation is given by $Z = f(x, y)$, and from below by plane area A, in the xy-plane bounded by a closed curve $y = g(x)$. To find the volume we divide the area A into small elements in the form of rectangles with dimensions δx , δy, then the elementary area is



$$\delta A_k = \delta x_k \, \delta y_k$$

Let $P(x_k, y_k)$ be any point in the elementary area δA_k, then $f(x_k, y_k)\delta A_k$ gives the volume of the column above δA as in the fig. Adding all such columns we obtain approximately the required volume

$$V \approx \sum f(x_k, y_k)\delta A_k$$

If we take the limit as $\delta A_k \to 0$, we obtain the required volume with great accuracy. Thus

$$V = \lim_{\delta A_k \to 0} \sum f(x_k, y_k)\delta A_k$$

This limit gives the double integral of the function $f(x, y)$ on the area A which is equal to the volume of the solid .

$$V = \iint_A f(x, y)dxdy = \iint_A z dxdy$$

EXAMPLE 1►

Find the volume bounded by the planes:x = 1 , x = 3 , y = 1 , y = 2 , z = 0 and the surface $z = x^2y^2$

SOLUTION

$$V = \int_{y=1}^{2}\int_{x=1}^{3} z\,d\,x\,d\,y = \int_{y=1}^{2} dy\int_{1}^{3} x^2y^2dx$$

$$= \int_1^2 y^2\left[\frac{x^3}{3}\right]_1^3 dx = \int_1^2 (9-\frac{1}{3})y^2dy$$

$$= \frac{26}{3}\left[\frac{y^3}{3}\right]_1^2 = \frac{26}{9}(8-1) = \frac{182}{9}.$$

EXAMPLE 2►

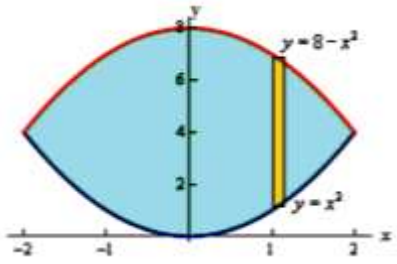
Find the volume of the solid that lies below the surface given by $z = 16xy + 200$and lies above the region in the xy-plane bounded by $y = x^2$ and $y = 8 - x^2$.

SOLUTION

$$\boldsymbol{V} = \iiint\ dzdxdy$$

$$= \iint_{R_{xy}} \left(\int_0^{z=200+16xy} dz\right) dxdy$$

$$V = \iint_{R_{xy}} (200+16xy)dxdy$$



Region in the xy-plane :

To obtain the points of intersection of the two curves, we solve the equations:$y = x^2$ and $y = 8 - x^2$

$$8 - x^2 = x^2 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

$$V = \int_{-2}^{2} dx \int_{x^2}^{8-x^2} (200 + 16xy)\, dy$$

$$V = \int_{-2}^{2} dx \left[200y + 8xy^2 \right]_{x^2}^{8-x^2}$$

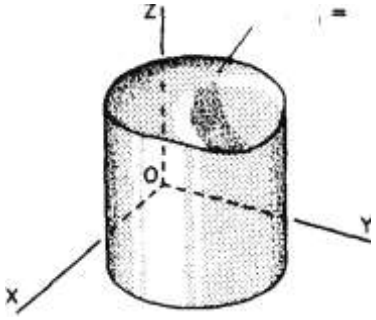
$$= \int_{-2}^{2} dx \left[200\left(8 - x^2 - x^2\right) + 8x\left(8 - x^2\right)^2 - 8x^5 \right]$$

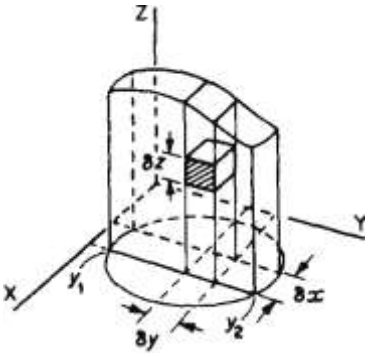
$$= \int_{-2}^{2} \left[1600 - 400x^2 + 512x - 128x^3 \right] dx$$

$$= \left[1600x - \frac{400}{3}x^3 + 256x^2 - 32x^4 \right]_{-2}^{2} = \frac{12800}{3}$$

◙DETERMINATION OF VOLUMES BY TRIPLE INTEGRAL►

Surface z, = f(x, y)





The element of volume δ v = δ x. δ y. δ z.
Summing the elements up the column, we have

$$\delta V = \sum_{z=0}^{z=z_1} \delta x\, \delta y\, \delta z$$

If we now sum the columns between y = y_1 and y = y_2 , we obtain the volume of the segment.

$$\delta V_s = \sum_{y=y_1}^{y=y_2} \sum_{z=0}^{z=z_1} \delta x \, \delta y \, \delta z$$

Then, summing all slices between x = x_1 and x = x_2, we have the total volume.

$$\delta V = \sum_{x=x_1}^{x=x_2} \sum_{y=y_1}^{y=y_2} \sum_{z=0}^{z=z_1} \delta x \, \delta y \, \delta z$$

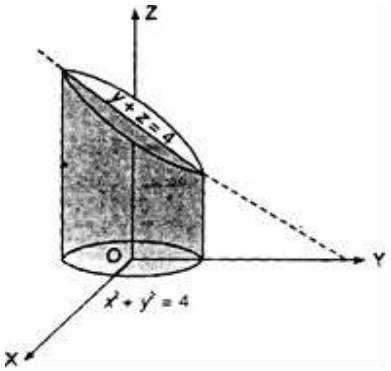
Then, as usual, if $\delta x \to 0, \delta y \to 0$ and $\delta z \to 0$, the resulting summation will be transformed into an integration in the form

$$V = \int_{x=x_1}^{x=x_2} \int_{y_1}^{y_2} \int_{0}^{z_1} dx \, d\, y \, d\, z$$

The result this time is a triple integral, but the development is very much the same as in our previous examples.

EXAMPLE 1►

Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes y + z = 4 and z= 0.



SOLUTION.

$x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4 - x^2}$

y + z =4 or z = 4 - y and z = 0

x varies from - 2 to + 2

$V = \iiint dxdydz$

$$V = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \int_{0}^{4-y} dz$$

$$V = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\, dy = \int_{-2}^{2} dx \left[4y - \frac{y^2}{2} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$$

$$= \int_{-2}^{2} dx \left[8\sqrt{4-x^2} \right] = 8 \int_{-2}^{2} \sqrt{4-x^2}\ dx$$

To calculate $\int_{-2}^{2} \sqrt{4-x^2}\ dx$, we use the substitution

$x = 2 \cos \theta$, $dx = -2 \sin \theta\, d\theta$

at $x = -2 \Rightarrow \theta = \pi$ and $x = 2 \Rightarrow \theta = 0$

The integral transforms to:

$$\int_{-2}^{2} \sqrt{4-x^2}\ dx = \int_{\pi}^{0} 2 \sin\theta \times (-2 \sin\theta d\theta)$$

$$= 2 \int_{0}^{\pi} (1 - cos2\theta)\, d\theta = 2\pi$$

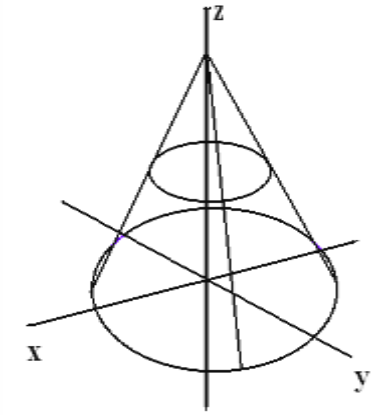
Hence $V = 16\pi$

EXAMPLE 2►

Set up the integral to find the volume of the solid that lies below the cone $z = 4 - \sqrt{x^2 + y^2}$ and above the xy-plane.

SOLUTION

The cone is sketched below



We can see that the region R is the blue circle in the xy-plane. We can find the equation by setting z = 0.

$$0 = 4 - \sqrt{x^2 + y^2}$$

Solving for y (by moving the square root to the left hand side, squaring both sides, etc) gives

$$y = \pm\sqrt{16 - x^2}$$

The "-" gives the lower limit and the "+" gives the upper limit.
For the outer limits, we can see that: $-4 \leq x \leq 4$

Putting this all together gives

$$V = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \left(4 - \sqrt{x^2 + y^2}\right) dxdy \;=\; 64\,\pi/3$$

Notice that this agrees with the formula
Volume = $\pi r^2 h/3$

EXAMPLE3►

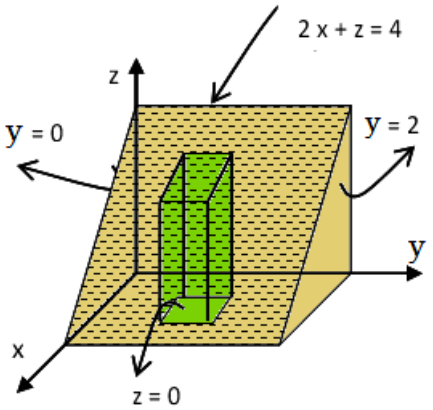
Calculate the integral $I = \iiint_v (2xy + 3xy^2 + 2z)\,dxdydz$, over the Region bounded by the planes $x = 0, y = 0, z = 0$

$$y = 2,\ 2x + z = 4$$

SOLUTION

$$I = \iiint_{v} (2xy + 3xy^2 + 2z)\,dxdydz$$

$$I = \iint_{R_{xy}} dxdy \int_{z=0}^{z=4-2x} (2xy + 3xy^2 + 2z)\,dz$$



$$= \iint_{R_{xy}} dxdy\ [2xyz + 3xy^2 z + z^2]_0^{z=4-2x}$$

$$= \iint_{R_{xy}} dxdy\ [4xy(2-x) + 6xy^2(2-x) + 4(2-x)^2]$$

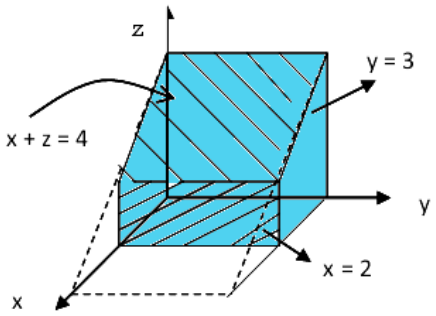
$$= \int_0^2 dy \int_0^2 [16 - 16x + 4x^2 + 8xy - 4x^2 y + 12xy^2 - 6x^2 y^2]\,dx$$

$$= \int_0^2 dy\,[16x - 8x^2 + 4\frac{x^3}{3} + 4x^2 y - 4\frac{x^3}{3} y + 6x^2 y^2 - 2x^3 y^2]_0^2$$

$$I=\int_0^2[\frac{32}{3}+\frac{16}{3}y+8y^2]dy=[\frac{32}{3}y+\frac{8}{3}y^2+\frac{8y^3}{3}]_0^2=\frac{160}{3}\#$$

EXAMPLE4►

Calculate the integral $\iiint_V x^2 y\,z\,dV$, where V is the region bounded by $x=2, y=3, z=-x+4$ and the coordinate planes



SOLUTION

$$I=\int_0^2 dx\int_0^3 dy\int_{z=0}^{4-x} x^2yz\,dz=\int_0^2 x^2dx\int_0^3 y\,dy\int_{z=0}^{4-x} z\,dz$$

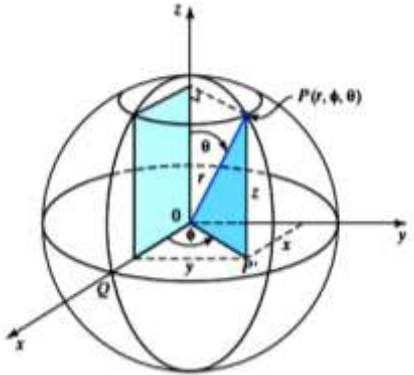
$$=\int_0^2 x^2dx\int_0^3 y\,dy\left(\frac{z^2}{2}\right)_{z=0}^{4-x}=\frac{1}{2}\int_0^2 x^2(4-x)^2\,dx\int_0^3 y\,dy$$

$$=\frac{1}{4}\int_0^2\left(16x^2-8x^3+x^4\right)dx\left(y^2\right)_0^3$$

$$=\frac{9}{4}\left(\frac{16}{3}x^3-2x^4+\frac{1}{5}x^5\right)_0^2$$

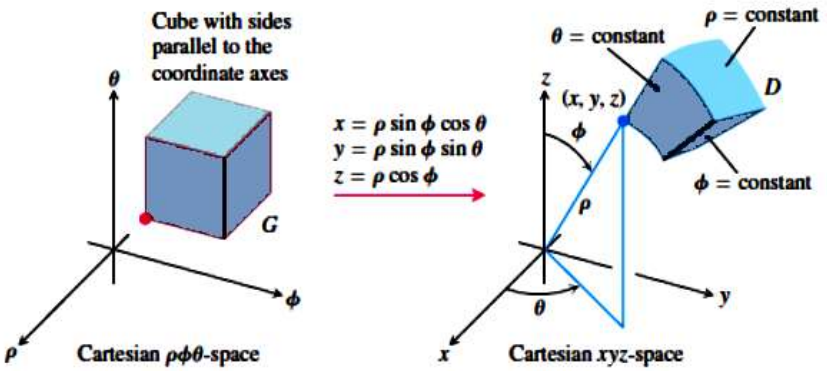
■SPHERICAL POLAR COORDINATES (r, φ, θ)

The spherical polar coordinate system (r, φ, θ) shown in Fig.



1is related to Cartesiancoordinates by
x = r sin θ cosφ,
y = r sin θ sin φ,
z = r cosθ,

with $0 \le \theta \le \pi$, $0 \le \varphi < 2\pi$.



The Jacobian of the transformation is

$$J(r,\theta,\phi) = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial\phi \end{vmatrix}$$

$$= r^2 \sin\theta$$

Element of volume

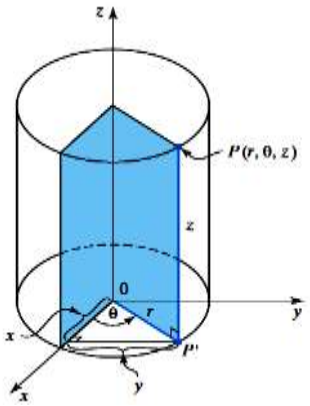
$$dV = dxdydz = r^2 \sin\theta dr\, d\theta\, d\phi$$

Element of surface area is

$$dS = R^2 \sin\theta d\theta d\phi$$

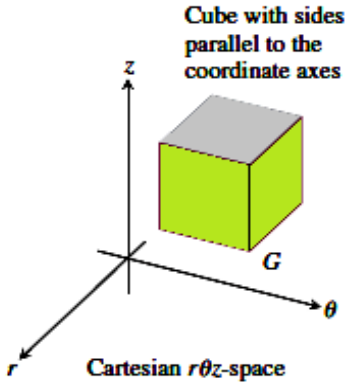
Cylindrical Polar Coordinates (ρ, θ, z)

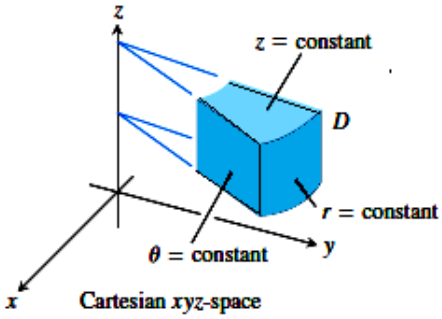
The cylindrical polar coordinate system (ρ, θ, z) is illustrated in the Fig., and itsrelationships to Cartesian coordinates is given by



$x = \rho \cos\theta$, $y = \rho \sin\theta$, $z = z$, with

$0 \le \theta < 2\pi$ and $\rho \ge 0$.





The Jacobian of the transformation is

$$J(\rho,\theta,z) = \begin{vmatrix} \partial x/\partial\rho & \partial x/\partial\theta & \partial x/\partial z \\ \partial y/\partial\rho & \partial y/\partial\theta & \partial y/\partial z \\ \partial z/\partial\rho & \partial z/\partial\theta & \partial z/\partial z \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$

Element of volume : $dV = dxdydz = \rho d\rho d\theta dz$

Element of surface area is: $dS = \rho d\theta dz$

EXAMPLE 1►

Find by triple integration the volume of a solid bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the paraboloid $x^2 + y^2 = 3z$

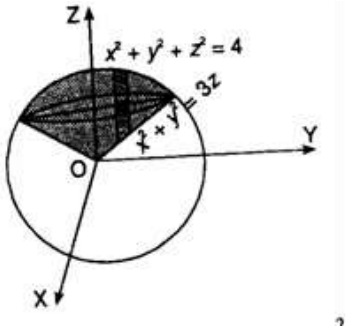
SOLUTION

$$x^2 + y^2 + z^2 = 4 \Rightarrow z = \sqrt{4 - x^2 - y^2} \quad (1)$$

$$x^2 + y^2 = 3z \Rightarrow z = \frac{x^2 + y^2}{3} \qquad (2)$$

Limits of z are

$$z = \sqrt{4 - x^2 - y^2} \text{ and } z = \frac{x^2 + y^2}{3}$$



Area of cross section at the intersection of (1) and (2) is given by equating

$$z = \sqrt{4 - x^2 - y^2} \text{ and } z = \frac{x^2 + y^2}{3}$$

$$4 - x^2 - y^2 = \left(\frac{x^2 + y^2}{3}\right)^2$$

$\Rightarrow (x^2 + y^2)^2 + 9(x^2 + y^2) - 36 = 0$

$\Rightarrow (x^2 + y^2 - 3)(x^2 + y^2 + 12) = 0$

The second factor is rejected , so the area of cross section is the circle $x^2 + y^2 = 3$,

$$V = \iiint dxdydz$$

$$V = \int dx\, dy \int_{\frac{x^2+y^2}{3}}^{\sqrt{4-x^2-y^2}} dz = \int dx\, dy \left(\sqrt{4 - x^2 - y^2} - \frac{x^2 + y^2}{3} \right)$$

To calculate the integral transform to polar coordinates (r , θ), since the cross section is the circle $x^2 + y^2 = 3$

Let $x = r \cos\theta$, $y = r \sin\theta$, $dx\, dy = r\, d\, r\, d\, \theta$

We get

$$V = \int_0^{2\pi} \int_0^{\sqrt{3}} r dr\, d\theta \left(\sqrt{4 - r^2} - \frac{r^2}{3} \right)$$

$$= \int_0^{2\pi} d\theta \left\{ \int_0^{\sqrt{3}} r\sqrt{4 - r^2}\, dr - \int_0^{\sqrt{3}} \left(\frac{r^3}{3} \right) dr \right\}$$

$$= 2\pi \left\{ \frac{-1}{2} \int_0^{\sqrt{3}} \sqrt{4 - r^2}\, (-2rdr) - \int_0^{\sqrt{3}} \left(\frac{r^3}{3} \right) dr \right\}$$

$$= 2\pi \left\{ \frac{-1}{2} \frac{\left(4 - r^2\right)^{3/2}}{\left(\frac{3}{2}\right)} - \left(\frac{r^4}{12} \right) \right\}_0^{\sqrt{3}}$$

$$= -2\pi \left\{ \left[\frac{1}{2} \frac{(4-3)^{3/2}}{\left(\frac{3}{2}\right)} + \left(\frac{9}{12}\right) \right] - \left[\frac{1}{2} \frac{(4-0)^{3/2}}{\left(\frac{3}{2}\right)} + (0) \right] \right\}$$

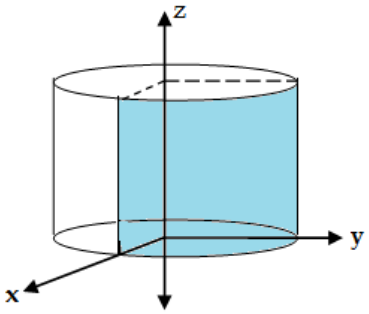
$$= -2\pi \left[\frac{1}{3} + \frac{3}{4} - \frac{8}{3} \right] = \frac{19\pi}{6}$$

EXAMPLE 2►

Calculate the integral

$$I = \frac{1}{2} \iint_S x\ y\ z\ (x+y)\, dS\ ,$$

where S is the part of the cylinder $x^2 + y^2 = 4$, which lies in the first octant and bounded by the planes: $z = 0, z = 5$



SOLUTION

$$I = \frac{1}{2} \iint_S x\ y\ z\ (x+y)\, dS$$

Transform to cylindrical coordinates, we have

$$z = z \quad y = \rho \sin\varphi \quad x = \rho \cos\varphi$$

$$dS = \rho\, d\varphi\, d\, z$$

$$\varphi : 0 \to \frac{\pi}{2} \quad , \quad z : 0 \to 5 \quad , \quad \rho = 2 = \text{constant}$$

$$I = \frac{1}{2} \iint_S \left(\rho^2 \cos^2\phi \cdot \rho \sin\phi \cdot z \ + \rho \cos\phi \cdot \rho^2 \sin^2\phi \cdot z \ \right) \rho\, d\, \phi dz$$

$$= \left(\frac{1}{2}\right)(2)^4 \iint_S \left[\cos^2\phi \sin\phi \cdot z + \cos\phi \sin^2\phi \cdot z\right] d\phi\, dz$$

$$I = 8\left[\int_0^5 z\, dz \int_0^{\frac{\pi}{2}} \cos^2\phi \sin\phi\, d\phi + \int_0^5 z\, dz \int_0^{\frac{\pi}{2}} \sin^2\phi \cos\phi\, d\phi\right]$$

$$= 8\left[\left(\frac{z^2}{2}\right)_0^5 \left(\frac{-\cos^3\phi}{3}\right)_0^{\frac{\pi}{2}} + \left(\frac{z^2}{2}\right)_0^5 \left(\frac{\sin^3\phi}{3}\right)_0^{\frac{\pi}{2}}\right]$$

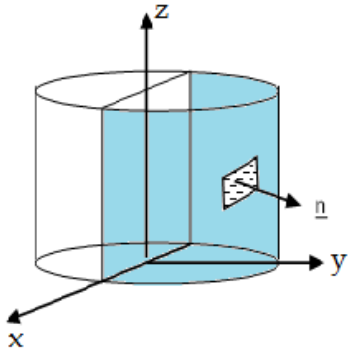
$$= 8\left[\left(\frac{25}{2}\right)\left(\frac{0+1}{3}\right) + \left(\frac{25}{2}\right)\left(\frac{1-0}{3}\right)\right] = \frac{200}{3}$$

EXAMPLE 3►

Calculate the integral

$I = \iint_S y^2 dS$, where S is the part of the cylinder $x^2 + y^2 = 4$, which lies in the first and second octants and bounded by the planes:

$y = 0$, $z = 0$, $z = 5$.



SOLUTION

$$I = \iint_S y^2 dS$$

Using cylindrical coordinates

$x = 2\cos\varphi,\ y = 2\sin\varphi\ ,\ z = z\ ,dS = 2d\varphi\, dz$

$$I = \iint_S y^2\, dS = \iint_S 4\sin^2\phi \cdot 2d\,\phi d\, z$$

$$\varphi = 0 \to \pi\ ,\ z: 0 \to 5$$

$$I = 4\int_0^5 d\, z \int_0^{\pi} (1-\cos 2\phi) d\,\phi = 4(z)_0^5 \left(\varphi - \frac{1}{2}\sin 2\varphi\right)_0^{\pi} = 20\pi$$ ■

EXERCISE 4.2

Transform each of the following integrals to the polar form, the calculate it

1► $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx$ 2► $\int_0^a\int_0^{\sqrt{a^2-y^2}} (x^2+y^2)dxdy$

3► $\int_0^{a/\sqrt{2}}\int_y^{\sqrt{a^2-y^2}} xdxdy$ 4► $\int_0^2\int_0^x ydydx$

5► $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} x^2 dydx$ 6► $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy$

7► Find the volume of a solid whose base is in the xy – plane anddetermined by the parabola $y = 4 - x^2$, and the line y =3x, and its upper surface is the plane z = x + 4.

8►Find the volume of a solid whose base is in the xy – plane and determined by the circle $x^2 + y^2 = a^2$, and its upper surface is the paraboloid $az = x^2 + y^2$.

9►Determine the integral $\iint_S xdxdy$, where S is the region enclosed by the line joining the two points (0 , 2) and (2 , 0) and the an arc from the circle $x^2 + (y-1)^2 = 1$

10►Determine the integral $\iint_S e^{x/y}$, where S is the region enclosed by the parabola $y^2 = x$, the two lines y = 1 and x = 0.

11► Calculate $\iint_S \frac{xdxdy}{x^2 + y^2}$, where S is the region enclosed by the parabola $y = \frac{1}{2}x^2$, the line y = x.

12►Calculate the integral $I = \iiint \phi \, dv$, over the upper hemi-sphere $x^2 + y^2 + z^2 = 16$, where $\phi = z\ x$.

Evaluate the following integrals

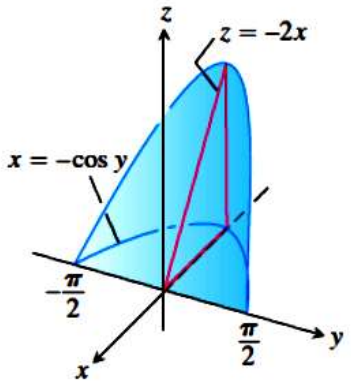
13. $\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \cos(x + y + z)dxdydz$

14. $\int_{\ln 6}^{\ln 7}\int_0^{\ln 2}\int_{\ln 4}^{\ln 5} e^{(x+y+z)}dxdydz$

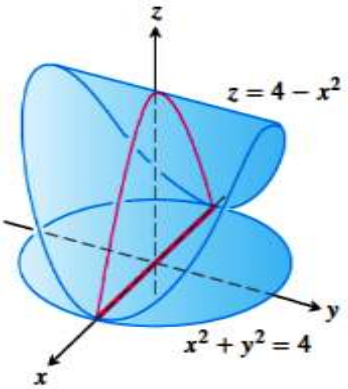
15. $\int_0^1\int_0^{x^2}\int_o^{x+y} (2x - y - z)dzdydx$

16. $\int_0^e\int_1^x\int_o^z \frac{2y}{z^3}dydzdx$

17► Find the volume of the wedge – shaped region enclosed on the side by the cylinder $x = -\cos y, -\pi/2 \le y \le \pi/2$, on the top by the plane $z = -2x$, and below by the xy-plane.



18. Find the volume of the solid that is bounded above bythe cylinder $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by the *xy*-plane.



19. Convertto **(a)** rectangular coordinates with the order of integration*dz dx dy*and**(b)** spherical coordinates. Then **(c)** evaluate one ofthe integrals.

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3\,dz\; r\,dr\,d\theta$$

20. (a) Convert to cylindrical coordinates. Then,
(b) evaluate the new integral.

$$\int_{0}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-(x^2+y^2)}^{(x^2+y^2)} 21x\, y^2\, dz\, dy\, dx$$

21. (a) Convert to spherical coordinates. Then,
(b) evaluate the new integral.

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} dz\, dy\, dx$$